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Interpolation at a Few Points

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For $k=2$ and 3, B. Shekhtman proved that $n+k-1$ is the smallest dimension of a subspace, $F \subseteq C(\mathbf{R}^n)$ that can interpolate to k specified real values at k distinct points in $\mathbf{R}^n$. Here we characterize such spaces that interpolate at a few points. The characterization provides an economical proof of Shekhtman's theorems, as well as establishing new properties of these spaces. © 1999 Academic Press

1. INTRODUCTION

The classical Haar spaces of approximation theory are the m-dimensional, m-interpolating subspaces of $C(X)$. For compact X, these are exactly the finite-dimensional spaces that admit unique best approximations to each $f \in C(X)$ (the Hausdorff–Young Theorem). But non-trivial (i.e., $m > 1$) Haar spaces only exist in $C(X)$ if X is homeomorphic to a subset $\mathbf{S}^1$, the circle (Mairhuber's Theorem). For example, it follows that a k-interpolating subspaces defined on $\mathbf{B}^n$, the open unit disc in $\mathbf{R}^2$, must have dimension at least $k+1$. Shekhtman's theorems give the minimal dimension, $i(n, k)$, of a k-interpolating subspace of $C(\mathbf{R}^n)$ for $k=2$ and 3.

This paper has two objectives: (i) to give short, basic proofs of Shekhtman's theorems, and (ii) to prove and apply a characterization theorem for interpolating spaces. We show that Shekhtman's theorems follow readily from (in fact, they are equivalent to) a form of the invariance of dimension theorem (i.e., $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$). Sections 2 and 3 list equivalent forms of k-interpolating spaces and the invariance of dimension principle. They result in economical proofs of Shekhtman's theorems in Section 4.

Section 5 characterizes the 2-interpolating spaces. It shows that an example Shekhtman found is, in a sense, the only type of 2-interpolating space. The characterization shows minimal 2-interpolating spaces in $C(\mathbf{B}^n)$ satisfy a

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maximum modulus principle. That is, a function in a minimal 2-interpolating space must be a constant function if it attains a local maximum or minimum. The characterization applies to interpolating subspaces defined on general domains. It shows, for example, that the minimal dimension of a 2-interpolating subspace defined on $\mathbf{S}^{n-1}$, the boundary of $\mathbf{B}^n$, is $n+1$.

A final Notes section contains comparisons to the literature.

2. PRELIMINARY LEMMAS

In this section we develop equivalent formulations of k-interpolating, and their consequences. Most of these follow readily from the results proceeding them and from definitions.

NOTATION AND DEFINITIONS 2.1. Throughout this paper X will be a Hausdorff topological space, and $C(X)$ will represent the continuous real-valued functions defined on X. $F \subseteq C(X)$ is k-interpolating if for each set of k distinct points $\{x_1, ..., x_k\} \subseteq X$ and k real numbers $\{\alpha_1, ..., \alpha_k\}$, there is an $f \in F$ such that $f(x_i) = \alpha_i$. In this paper F will be a linear space. The linear span of $\{f_1, ..., f_p\} \subseteq C(X)$ is denoted by span $\{f_1, ..., f_p\}$. For a point $x \in X$ the associated point evaluation functional, $\hat{x}$, is defined by $\hat{x}(f) = f(x)$ for all $f \in C(X)$. The sign of a real number α is written sgn α. The vector space of n-tuples of real numbers is written $\mathbf{R}^n$, and the jth coordinate projection, π_j is $\pi_j((x_1, ..., x_n)) = x_j$.

LEMMA 2.2. *F is k-interpolating if and only if for any set of k points $\{x_1, x_2, ..., x_k\}$ in X there is an f in F, such that $f(x_i) = 0$ for $i = 1, ..., k-1$ and $f(x_k) \neq 0$.*

LEMMA 2.3. *F is k-interpolating if and only if for any set of k points, $\{x_1, x_2, ..., x_k\}$, in X and any sequence of k plus or minus signs, $\{s_1, s_2, ..., s_k\}$, there exists an f in F, such that* sgn $f(x_i) = s_i$, *for $i = 1, ..., k$.*

Proof. Suppose that for any such sequence of signs, a member of F has such a prescribed sequence of sign changes. Zero is the only linear combination of $\{\hat{x}_1 ..., \hat{x}_k\}$ that annihilates F. Hence they are linear independent as functionals defined on F. It follow that for any $1 \leqslant j \leqslant k$ there is a function in F that vanishes at x_i for $i \neq j$, but does not vanish at x_j. (For example, this follows from the Hahn–Banach Theorem.) ∎

LEMMA 2.4. *$F = \text{span}\{1, f_1, ..., f_p\}$ is k-interpolating if and only if for any set of k points $\{x_1, x_2, ..., x_k\}$ in X there is an f in* span$\{f_1, ..., f_p\}$, *such that $f(x_1) = f(x_2) = \cdots = f(x_{k-1}) \neq f(x_k)$.*

COROLLARY 2.5. *If* span$\{1, f_1, ..., f_p\}$ *is k-interpolating, then* span$\{f_1, ..., f_p\}$ *is* $k-1$*-interpolating on* $X-Z$, *where* $Z=\{x \in X\colon f_i(x)=0, \text{ for } i=1, ..., p\}$.

Proof. Use Lemma 2.4 to show that span$\{f_1, ..., f_p\}$ satisfies the conditions in Lemma 2.2.

DEFINITION 2.6. $F \subseteq C(X)$ is point separating if for two distinct points x and y in X there is an $f \in F$ such that $f(x) \neq f(y)$.

LEMMA 2.7. *If F is k-interpolating for* $k \geqslant 2$, *then F is both* $k-1$*-interpolating and point separating.*

LEMMA 2.8. *If* span$\{1, f_1, ..., f_p\}$ *is k-interpolating, then* span$\{f_1, ..., f_p\}$ *is point separating.*

LEMMA 2.9. Span$\{1, f_1, ..., f_p\}$ *is 2-interpolating, if and only if* span$\{f_1, ..., f_p\}$ *is point separating.*

LEMMA 2.10. *Let Y be a subset of* $\mathbf{R}^n$ *that contains interior. The following are equivalent*:

(a) *there is a p-dimensional k-interpolating subspace defined on* $\mathbf{R}^n$,

(b) *there is a p-dimensional k-interpolating subspace defined on Y*,

(c) *there is a p-dimensional k-interpolating subspace defined on* $\mathbf{R}^n$ *that contains the constant functions*,

(d) *there is a p-dimensional k-interpolating subspace defined on Y that contains the constant functions.*

Proof. We need only observe that (b) implies (c). Let B be a neighborhood in Y on which there is function, f, in the interpolating space, F, that is positive. Let h be a homeomorphism of $\mathbf{R}^n$ into B. Then $\{(g/f)(h(\cdot))\colon g \in F\}$ is a k-interpolating subspace defined on $\mathbf{R}^n$ that contains the constant functions. ∎

LEMMA 2.11. *If F is a point separating n-dimensional subspace of* $C(X)$, *then there is a continuous, one-to-one mapping of X onto a subset of* $\mathbf{R}^n$.

Proof. If $\{f_1, ..., f_n\}$ is a basis for F, $(f_1(x), ..., f_n(x))$ is the desired mapping.

3. INVARIANCE OF DIMENSION

We will use the basic topological principle of invariance of dimension. From one of its classical forms we will adapt the forms best for our uses here.

THEOREM 3.1 (Invariance of Dimension). *A homeomorphism of* $\mathbf{R}^n$ *into* $\mathbf{R}^n$ *carries* $\mathbf{R}^n$ *onto an open set.*

COROLLARY 3.2. *A one-to-one, continuous mapping, f, of an open set* $Y \subseteq \mathbf{R}^n$ *into* $\mathbf{R}^n$ *is a homeomorphism of Y onto an open subset in* $\mathbf{R}^n$.

Proof. Suppose x_o is in Y. We want to show $f(Y)$ contains an $\mathbf{R}^n$-neighborhood of $f(x_o)$, on which f^{-1} is continuous. Choose $\varepsilon > 0$ so that N, the compact neighborhood of x_o of radius ε, is contained in Y. $\mathbf{R}^n$ is homeomorphic to the interior of N via a mapping such as $h(x) = ((x - x_o)/(1 + \|x - x_o\|))\, \varepsilon + x_o$. Since f is a homeomorphism on N, it is a homeomorphism on the interior of N. Since the range of f on the interior of N is the same set as the range of the composite of f and h on $\mathbf{R}^n$, it is an open set in $\mathbf{R}^n$. ∎

COROLLARY 3.3. *If there is a one-to-one, continuous mapping, f, of an open set* $Y \subseteq \mathbf{R}^n$ *into* $\mathbf{R}^m$, *then* $m \geqslant n$.

4. SHEKHTMAN'S THEOREMS

In this section we use the principle of invariance of dimension and our rephrasings of k-interpolating to reprove three of Shekhtman's results.

COROLLARY 4.1. *If X contains an open subset of* $\mathbf{R}^n$ *and* $F \subseteq C(X)$ *is point separating, then* $\dim F \geqslant n$.

Proof. This follows from the Lemma 2.11 and the invariance of dimension corollaries.

COROLLARY 4.2 ([7]). *If X contains an open subset of an infinite dimensional topological linear space, then C(X) does not contain a finite-dimensional point separating subspace.*

LEMMA 4.3. *There does not exist an n-dimensional 2-interpolating subspace of* $C(\mathbf{R}^n)$.

Proof. From Lemmas 2.10, 2.8 and Corollary 4.1, such a subspace would contradict the invariance of dimension corollary.

EXAMPLE 4.4. Span$\{1, \pi_1, ..., \pi_n\}$ is 2-interpolating on $\mathbf{R}^n$.

Proof. $\{\pi_1, ..., \pi_n\}$ separates the points of $\mathbf{R}^n$ so Lemma 2.9. applies.

Notation 4.5. $i(n, k) = \min\{\dim F\colon F \subseteq C(\mathbf{R}^n)$, and F is k-interpolating$\}$.

THEOREM 4.6 ([7]). $i(n, 2) = n+1$.

Proof. This follows from 4.3 and 4.4.

LEMMA 4.7. *If $C(\mathbf{R}^n)$ contains a p-dimensional k-interpolating subspace, F, then it contains a $p-1$-dimensional $k-1$-interpolating subspace.*

Proof. For any $x \in \mathbf{R}^n$, $\{f \in F\colon f(x) = 0\}$ is a $p-1$-dimensional $k-1$-interpolating subspace on the open set $\mathbf{R}^n \backslash \{x\}$. Hence, by Lemma 2.10, there exist such a space defined on all of $\mathbf{R}^n$. ∎

EXAMPLE 4.8. Span$\{1, \pi_1, ..., \pi_n, \sum_{i=1}^n \pi_i^2\}$ is 3-interpolating on $\mathbf{R}^n$.

Proof. Let $x, y, z \in \mathbf{R}^n$. We will give a geometric proof that there is a function in the span that vanishes at x and y, but not at z. By the Lemmas 2.2, this will prove the space is 3-interpolating. We make two observations. First, no three points on the graph of the n-dimensional parabola $p = \sum_{i=1}^n \pi_i^2$ are on a straight line. In particular, $(z, p(z))$ is not on the line containing $(x, p(x))$ and $(y, p(y))$. Second, the line in $\mathbf{R}^n \times \mathbf{R}^1$ containing $(x, p(x))$ and $(y, p(y))$ is the intersection of the graphs of the affine functions defined on $\mathbf{R}^n$ which contain both $(x, p(x))$ and $(y, p(y))$. The affine functions are precisely the functions in span$\{1, \pi_1, ..., \pi_n\}$. Hence there is a function $g \in$ span$\{1, \pi_1, ..., \pi_n\}$, such that $g(x) = p(x)$, and $g(y) = p(y)$, but $g(z) \neq p(z)$. So $g - p$ is the function we sought to produce. ∎

Comment. The only property of $\sum_{i=1}^n \pi_i^2$ that we used in the proof was that no three points on its graph are in a straight line. So we could replace $\sum_{i=1}^n \pi_i^2$ with any function possessing that property.

This example and results 4.6 and 4.7 prove:

THEOREM 4.9 ([7]). $i(n, 3) = n+2$.

5. CHARACTERIZATION

DEFINITION 5.1. We will call a mapping of X into Y an injection if it is both one-to-one and continuous.

THEOREM 5.2 (Characterization). *Let $F \subseteq C(X)$ be n-dimensional (or $(n+1)$-dimensional and contain the constants, resp.). Then F is:*

(a) *point separating if and only if there is an injection, h, of X into $\mathbf{R}^n$ such that $F = \{f(h(\cdot)) : f \in \operatorname{span}\{\pi_1, ..., \pi_n\}\}$ (or $F = \{f(h(\cdot)) : f \in \operatorname{span}\{1, \pi_1, ..., \pi_n\}\}$, resp.), and*

(b) *k-interpolating if and only if, in addition, $\{h(x_i)\}_{i=1}^k$ are linearly independent in $\mathbf{R}^n$ whenever $\{x_i\}_{i=1}^k$ are distinct points in X.*

Proof. The idea for the proof for both parts is the same. For $x \in X$, let $h(x)$ be the member of the dual space of F obtained by restricting $\hat{x}$, the x-point evaluation functional, to F. The lemma follows from the observations that: (i) F^*, the dual of F, is homeomorphic to $\mathbf{R}^n$, (ii) F is the space of all linear functions defined on F^*, and (iii) h is an injection. The proofs for the cases when $1 \in F$ reduce to the proven ones. Lemma 2.7 is used for part (b). ∎

COROLLARY 5.3 (2-Interpolation Characterization). (a) *F is an n-dimensional 2-interpolating subspace of $C(X)$ if and only if there is an injection, h, of X into $\mathbf{R}^n$ such that:*

(i) *$F = \{f(h(\cdot)) : f \in \operatorname{span}\{\pi_1, ..., \pi_n\}\}$, and*

(ii) *there does not exist distinct x and y in X and $\alpha \in \mathbf{R}^n$ such that $h(x) = \alpha h(y)$.*

(b) *F contains the constants and is an $(n+1)$-dimensional 2-interpolating subspace of $C(X)$ if and only if there is an injection h of X into $\mathbf{R}^n$ such that $F = \{f(h(\cdot)) : f \in \operatorname{span}\{1, \pi_1, ..., \pi_n\}\}$.*

Proof. Part (a) is a direct corollary to the characterization for k-interpolating spaces. The proof of part (b) uses Lemma 2.9.

COROLLARY 5.4. *With the setting of Theorem 5.2, if X is either compact, or an open subset of $\mathbf{R}^n$, then h is an homeomorphism.*

Notation 5.5. We use $\mathbf{S}^{n-1}$ and $\mathbf{B}^n$ to represent respectively $\{x \in \mathbf{R}^n : \|x\| = 1\}$ and $\{x \in \mathbf{R}^n : \|x\| < 1\}$.

COROLLARY 5.6. *The minimal dimension of a 2-interpolating, constant containing subspace of $C(\mathbf{S}^{n-1})$ is $n+1$.*

Proof. span$\{1, \pi_1, ..., \pi_n\}$ restricted to $\mathbf{S}^{n-1}$ is an $n+1$ dimensional 2-interpolating space. The other direction follows from the above corollaries 5.3 and 5.4 since $\mathbf{R}^{n-1}$ does not contain a homeomorphic copy of $\mathbf{S}^{n-1}$.

EXAMPLE 5.7. In general h may not have a continuous inverse. For example, let X be $\mathbf{B}^n$ but with the discrete topology. Let F be the linear functionals on $\mathbf{R}^n$ restricted to X, then although F is a point separating subspace of $C(X)$ there is no homeomorphism of X into $\mathbf{R}^n$.

COROLLARY 5.8. *Let F be an $n+1$-dimensional subspace of functions defined on an open set $X \subseteq \mathbf{R}^n$ that is 2-interpolating and contains the constants. If f is a non-constant function in F, then: (i, maximum modulus) f does not have a local maximum or minimum in X, and (ii, dimension of level sets) a component, K, of a non-trivial level set for f is homeomorphic to an open subset of $\mathbf{R}^{n-1}$.*

Proof. This follows from the characterizations 5.3(b), The principle of invariance of dimension 3.2, and the equivalent properties for span$\{\pi_1, ..., \pi_n\}$.

6. POSITIVE FUNCTIONS

If a k-interpolating subspace $F \subseteq C(X)$ contains a positive function p, then $\{f/p : f \in F\}$ is a k-interpolating subspace that contains the constants. The characterization of constant containing k-interpolating spaces have injections into domains one dimension lower then for interpolating spaces without constants. However, we show below that not all interpolating spaces contain a positive function.

Notation 6.1. Let F be an n-dimensional point-separating subspace of $C(X)$. Let h be the injection of the characterization lemmas, and let $Y = h(X)$.

LEMMA 6.2. *F contains a positive function if and only if zero is not in the convex hull of Y.*

Proof. If 0 is not in the convex hull of Y, the Hahn–Banach theorem provides a linear functional, f, that separates 0 from Y. Hence $f(h(\cdot))$ (or its negative) is strictly positive on X. By the characterization lemmas it is also in F.

For the reverse implication, suppose that f is a linear functional defined on $\mathbf{R}^n$ such that $f(h(\cdot))$ is in F and is positive on X. If zero is in the convex hull of X, there are points $\{y_i\}_{i=1}^{p}$ such that zero is a convex combination of $\{y_i\}p_{i=1}$. But this is not possible since $f(y_i) > 0$ for each i. Hence f is strictly positive for every convex combination of them. ▮

PROPOSITION 6.3. *For each $n > 1$, there is an $n+1$-dimensional, 2-interpolating subspace $F \subseteq C(\mathbf{B}^n)$ that does not contain a positive function.*

Proof. We first construct such a subspace on a subset of $\mathbf{S}^n$. Let

$$A_1 = \left\{ (\cos\theta, \sin\theta, 0, ..., 0) : 0 \leqslant \theta \leqslant \frac{\pi}{2} \right\},$$

$$A_2 = \left\{ (\cos(\pi-\phi), \sin^2(\pi-\phi), \sin(\pi-\phi)\cos(\pi-\phi), 0, ..., 0) : \frac{\pi}{2} \leqslant \phi \leqslant \frac{3\pi}{4} \right\},$$

and

$$A_3 = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \sin\theta, \frac{1}{\sqrt{2}} \cos\theta, 0, ..., 0 \right) : \frac{\pi}{4} \leqslant \theta \leqslant \frac{5\pi}{4} \right\}.$$

Each of these sets is a path on $\mathbf{S}^n \subseteq \mathbf{R}^{n+1}$. That is, A_1 connects $(-1, 0, ..., 0)$ to $(0, 1, 0, ..., 0)$; A_2 begins at $(0, 1, 0, ..., 0)$ and terminates at $(1/\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 0, ..., 0)$; and A_3 starts at $(1/\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 0, ..., 0)$ and ends at $(1/\sqrt{2}, -\frac{1}{2}, -\frac{1}{2}, 0, ..., 0)$. Let $A = \bigcup_{i=1}^{3} A_i$. First we want to show that A does not contain a pair of antipodal points. Except for $(-1, 0, ..., 0)$, every point in $A_1 \cup A_2$ has a positive second coordinate. Hence, there is no pair of antipodal points in $A_1 \cup A_2$. Every point in A_3 has a positive first coordinate, so A_3 does not contain a pair of antipodal points. The remaining possibility is that there are points $a \in A_1 \cup A_2$ and $b \in A_3$, such that $a = -b$. This would imply that the first coordinate of a is $-1/\sqrt{2}$. The only such point is $(-1/\sqrt{2}, \frac{1}{2}, 0, ..., 0) \in A_1$, and the opposite point $(1/\sqrt{2}, \frac{1}{2}, -\frac{1}{2}, 0, ..., 0)$, is not in A. Since A contains the three points: $(-1, 0, ..., 0)$, $(1/\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 0, ..., 0)$, and $(1/\sqrt{2}, -\frac{1}{2}, -\frac{1}{2}, 0, ..., 0)$, we observe that 0 is in the convex hull of A.

Now let $d = \operatorname{dist}(A, -A)$. Since A and $-A$ are disjoint compact sets, $d > 0$. Let $Y = \{x \in \mathbf{S}^n : \operatorname{dist}(x, A) < d/3\}$. Let G be the linear functionals on $\mathbf{R}^{n+1}$. No function in G can be positive on Y, since that would imply the function were positive on the convex hull of Y and in particular at 0. Also G is 2-interpolating on Y, because we have constructed Y to have the property that every pair of points in Y are linearly independent (i.e., no point in Y is a multiple of another point in Y). Finally let h be a homeomorphism of $\mathbf{B}^n$ onto Y. Then $F=\{g(h(\cdot)) : g \in G\}$ is the desired $n+1$-dimensional, 2-interpolating subspace of $C(\mathbf{B}^n)$ that does not contain a positive function. ▮

7. NOTES

We used the invariance of dimension principle to prove that $i(n, 2)=n+1$. In fact, $i(n, 2)=n+1$ is equivalent to Corollary 3.3. That is, suppose that $i(n, 2)=n+1$, but that $h(x)=(f_1(x), ..., f_{n-1}(x))$ were an injection into $\mathbf{R}^{n-1}$. Then Lemma 2.9 would show that $\{1, f_1, ..., f_{n-1}\}$ were an n-dimensional, 2-interpolating space. There are two observations from this. First, a pure analysis proof that $i(n, 2)=n+1$ would provide a new entry to these topological results. For example, if $\{f_1, ..., f_n\}$ is point separating, than $\{f_1, ..., f_{n-1}\}$ is point separating on the level sets of f_n. So, if there were a non-topological proof of Corollary 5.8(ii, dimension of level sets), we could reduce the problem of showing that $i(n, 2)=n+1$ to the easily verifiable $i(2, 2)=3$, The second observation is that Shekhtman's theorem that $i(n, 2)=n+1$, must be as intricate as the basic but profound invariance of dimension principle.

Computing $i(2, k)$ is equivalent to finding the minimal p so that there are k-regular embeddings (i.e., k points in the range are linearly independent) of $\mathbf{R}^2$ into $\mathbf{R}^p$. Some marvelous bounds were found by Cohen and Handel (1978) using algebraic topological arguments.

In general $i(n, k)$ is not known. Even $i(2, 4)$ is not known.

Shekhtman uses separate arguments for computing $i(n, 2)$ and $i(n, 3)$. Here, we proved $i(n, 2)=n+1$ and we reduced the case for $k=3$ to this proven one.

Corollary 4.2 is a little stronger than Shekhtman's theorem. Here it is stated for a space that is point separating instead of 2-interpolating,

The example, 4.8, of a minimal 3-interpolating space is the same as Shekhtman's. Although the proof is new, the idea comes from reading Shekhtman's proof.

The subspaces, which contain no positive function, of Theorem 6.3 do not exist if $n=1$. This is a special case of results in [11]. In fact, Every n-dimensional, n-interpolating subspace of $C([-1, 1])$ contains a positive function. We present a simple proof for $n=2$. The example was shown to me by Allan Pinkus.

EXAMPLE. A 2-dimensional, 2-interpolating subspace of $F \subseteq C([-1, 1])$ contains a positive function.

Proof. Let g and h be in F have the properties that $g(-1)=h(1)=0$ and $g(1)=h(-1)=1$. Then $g+h$ is positive on $[-1, 1]$. ▮

Say that a space $M \subset C(X)$ has p-dimensional sets of best approximation if p is the maximum number such that there is a function f with the property that the span of the best approximations to f from M has dimension p. The connection of k-interpolating spaces to classical approximation theory is that an m-dimensional subspace is k-interpolating if and only if it admits $m-k$-dimensional sets of best approximations [8]. The Hausdorff–Young Theorem results when $k = m$.

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